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Ergodic and Quasideterministic Properties of Finite-Dimensional Stochastic Systems[1]

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The ergodic and stability properties of certain stochastic models are studied. Each model is described by a finite-dimensional stochastic process $x^\lambda(t)$ satisfying $dx^\lambda(t) = \mathscr{F}^\lambda(x^\lambda, t)\,dt + \lambda\,dz(t)$, where $\mathscr{F}^\lambda$ represents a "secular force" and $z(t)$ is a stochastic process with given statistical properties. Such a model may represent a reduced description of an infinite-particle system. Then $x^\lambda(t)$ may be either a set of macrovariables fluctuating about thermal equilibrium or the macrostate of a system maintained through pumping in a nonequilibrium state. Two Markovian models for which $z(t)$ is Wiener and $\mathscr{F}^\lambda(y, t) = G(\lambda, y(t))$ for some G nonlinear in $y(t)$ are shown to possess a unique stationary probability density which is approached by any other density as $t \to \infty$. For one of these models, which is of Hamiltonian type, the stationary state is given by the Maxwell–Boltzmann distribution. A particular form of non-Markovian model is also proved to have the above mixing property with respect to the Maxwell–Boltzmann distribution. Finally, the behavior of the sample paths of $x^\lambda(t)$ for small values of the parameter λ is investigated. In the case when $z(t)$ is Wiener and $\mathscr{F}^\lambda(y, t) = G(y(t))$, it is shown that $x^\lambda(t)$ will remain close to the deterministic trajectory $x^0(t)$ (corresponding to $\lambda = 0$) for all $t \geqslant 0$ if and only if $x^0(t)$ is highly stable with respect to small perturbations of the initial conditions.

KEY WORDS: Stochastic differential equations; Markovian and non-Markovian processes; mixing; quasideterministic behavior; finite stochastic system; quasideterministic stationary states; ergodic properties.

1. INTRODUCTION

In this paper a rigorous investigation will be made of the ergodic and "almost deterministic" behavior of certain stochastic processes. The processes considered will in each case lie in a finite-dimensional phase space X and satisfy a generalized Langevin-type equation of the form

$$dx^\lambda(t) = \mathscr{F}^\lambda(x^\lambda, t)\,dt + \lambda\,dz(t) \tag{1}$$

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Here $\lambda \geqslant 0$ is a dimensionless parameter, $\mathscr{F}^\lambda$ is an X-valued functional of the function x^λ and of the time t, which represents a "secular force" producing dissipation, and $z(t)$ is an X-valued stochastic process corresponding to a "fluctuating force" whose properties are specified statistically.

A stochastic system Σ described by the variables $x^\lambda(t)$ may sometimes be embedded as a subsystem of a larger mechanical system in which the time evolution is conservative. If (1) is a stochastic differential equation,[1,2] so that $x^\lambda(t)$ is Markovian, such an embedding may be achieved by means of a Nagy dilation (cf. Refs. 3 and 4). Alternatively, certain special mechanical models, such as, for example, that of Ford *et al.*,[5] have a contracted description in terms of an equation of the form (1).

More generally, but on a heuristic level, equations of stochastic type are frequently used to model the behavior of open systems, both near and far from equilibrium, which undergo fluctuating forces due to their thermal environment (see, e.g., Refs. 6–9). In this context the Σ variables may be taken to correspond to a set of macrovariables of some mechanistic system M, and the fluctuating forces, whose stochastic properties are usually conjectured on the basis of physical considerations, to arise from interactions with the large number of residual variables of M.

The present paper will be devoted to a study of two aspects of the behavior of $x^\lambda(t)$. The first concerns properties of an ergodic nature, which hold in the long-time limit. Let $m_{\lambda,t}$ denote the distribution of $x^\lambda(t)$ on X. In the existing literature it is often assumed without proof or justified only by nonrigorous arguments (cf. Refs. 7, 10, and 11) that $m_{\lambda,t}$ approaches some stationary limiting value as $t \to \infty$. In Propositions 3.1, 3.2, and 3.5 of Section 3 we give a rigorous proof that certain (both Markovian and non-Markovian) models exhibit this mixing behavior.

We study also the behavior of the sample paths of $x^\lambda(t)$ at finite times, and the extent to which this process remains "almost deterministic" when the strength of fluctuations, as governed by the parameter λ, is sufficiently small. Previous nonrigorous treatments of the master equation for a set of macrovariables have suggested a connection between such "quasideterministic" behavior of a stochastic system and stability with respect to initial conditions of the corresponding causal evolution obtained by neglecting fluctuations.[12–14] In Proposition 4.3 of Section 4 we place this on a rigorous footing for a particular Markovian model.

The models to be considered are described in Section 2. The following notation will be used. R, R^+, and R^n will denote the real line, the positive reals, and Euclidean n-dimensional space, respectively, and for $n \geqslant 1$ we shall write $|x|$ for the Euclidean norm of $x \in R^n$. If (X, σ, m) is a measure space, and $1 \leqslant p \leqslant \infty$, $L^p(X, m)$ will denote the L^p-class functions on X with respect to m, and 1_A the characteristic function of the set $A \in \sigma$. The L^p-class functions on R^n with respect to Lebesgue measure will be denoted by $L^p(R^n)$.

2. DESCRIPTION OF THE MODELS

We shall now formulate the equations of evolution of the three stochastic models, Σ_A, Σ_B, and Σ_C, with which we shall be concerned. Σ_A and Σ_B are described by Markovian diffusion processes, while Σ_C is non-Markovian.

Σ_A: $X = R^{2n}$, $n < \infty$. $x^\lambda(t) = (q^\lambda(t), p^\lambda(t))$ satisfies

$$\begin{aligned} dq^\lambda(t) &= p^\lambda(t)\,dt \\ dp^\lambda(t) &= [F(q^\lambda(t)) - \tfrac{1}{2}\lambda^2\beta p^\lambda(t)]\,dt + \lambda\,dw(t) \end{aligned} \tag{2}$$

where λ, $\beta > 0$, $w(t)$ is an R^n-valued Wiener process, and $F: R^n \to R^n$ is a (generally nonlinear) function satisfying a Lipschitz condition with constant k, i.e., $|F(q_1) - F(q_2)| \leqslant k|q_1 - q_2|$ for all $q_1, q_2 \in R^n$. We shall assume that $F = -\text{grad}\, V$ for some $V: R^n \to R$. With β equal to the inverse temperature, the Langevin equations describing the motion of a Brownian particle in an external force field according to the Ornstein–Uhlenbeck theory[15,16] have this form, as do the equations of the Ford–Kac–Mazur model in the limit as the number of oscillators constituting the heat bath becomes infinite.

Σ_B: $X = R^n$, $n < \infty$. $x^\lambda(t)$ satisfies

$$dx^\lambda(t) = F(x^\lambda(t))\,dt + \lambda\,dw(t) \tag{3}$$

where $\lambda > 0$ and F and $w(t)$ are defined as for Σ_A. The Langevin equation for a Brownian particle acted on by a velocity-dependent frictional force has this form, as does the Smoluchowski approximation to the equations of the Ornstein–Uhlenbeck theory for a Brownian particle in an external force field. Heuristically, such an equation may be used to describe not only equilibrium situations, but also systems far from equilibrium, as in the case of Haken's[7] classical treatment of the single-mode laser.

Σ_C: $X = R^{2n}$, $n < \infty$. $x^\lambda(t) = (q^\lambda(t), p^\lambda(t))$ satisfies

$$\begin{aligned} &dq^\lambda(t)/dt = p^\lambda(t) \\ &dp^\lambda(t)/dt = F(q^\lambda(t)) - 2\lambda^2\alpha\beta^{-1}\int_0^t e^{-\alpha(t-s)}p^\lambda(s)\,ds + 2\lambda\alpha\beta^{-1}\int_{-\infty}^t e^{-\alpha(t-s)}\,dw(s) \end{aligned} \tag{4}$$

Here λ, α, $\beta > 0$, F and $w(t)$ are defined as for Σ_A, and $\int_{-\infty}^t e^{-\alpha(t-s)}\,dw(s)$ is an Itô stochastic integral (see, e.g., Ref. 2).

The form of the dissipative part of the secular force in model Σ_C is a consequence of a fluctuation-dissipation type theorem, which is proved in Ref. 17, using methods similar to those of Ref. 18. This theorem is needed to ensure that, when F is linear, the effect of the fluctuations is to drive Σ_C into a terminal state which, in the weak coupling limit, becomes the canonical equilibrium state at inverse temperature β. Equations (2) are obtained from (4) in the limit as $\alpha \to \infty$.

3. MIXING

Let Σ be a stochastic model whose variables $x^\lambda(t)$ lie in a finite-dimensional phase space X and have the distribution $m_{\lambda,t}$ on X. If there exists a probability measure $\bar{m}_\lambda$ on X such that $\lim_{t\to\infty} m_{\lambda,t} = \bar{m}_\lambda$ in the weak topology of measures whenever $m_{\lambda,0}$ is absolutely continuous with respect to $\bar{m}_\lambda$, then Σ will be said to be mixing with respect to $\bar{m}_\lambda$. In this case the measure $\bar{m}_\lambda$ is clearly invariant under the evolution of $x^\lambda(t)$.

3.1. Markovian Models

Let $\mathscr{B}$ denote the sigma-algebra of Borel sets of X, and $B(X, \mathscr{B})$ the Banach space of bounded, measurable functions on X with the sup norm $\|\ \|_\infty$. Then for each of the models Σ_A and Σ_B we can define a positivity-preserving contraction semigroup $\{T_t\}$ on $B(X, \mathscr{B})$ by[19]

$$(T_t f)(x) = \int_X P(t, x, dy) f(y) \tag{5}$$

where $P(t, x, E)$, $t > 0$, $x \in X$, $E \in \mathscr{B}$, is the transition probability of the corresponding diffusion process $x^\lambda(t)$. Let $C_0(X)$ and $C^2_{\text{com}}(X)$ denote, respectively, the continuous functions vanishing at infinity and the twice continuously differentiable functions with compact support on X, both with the norm $\|\ \|_\infty$. The restriction of $\{T_t\}$ to $C_0(X)$ is a strongly continuous semigroup whose generator is an extension of the differential operator G defined on $C^2_{\text{com}}(X)$ by[15]

$$\begin{aligned}\Sigma_A:\quad G &= \frac{1}{2}\lambda^2 \exp\left(\frac{1}{2}\beta|p|^2\right) \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\left[\exp\left(-\frac{1}{2}\beta|p|^2\right)\frac{\partial}{\partial p_i}\right] \\ &\quad - \{H(q, p),\ \} \\ &= G_1 - G_2, \quad \text{say}\end{aligned}$$

where $H(q, p) = V(q) + \frac{1}{2}|p^2|$, and $\{\ \}$ is a Poisson bracket.

$$\Sigma_B:\quad G = \sum_{i=1}^{n}\left[\frac{1}{2}\lambda^2 \frac{\partial^2}{\partial x_i^2} - \frac{\partial V}{\partial x_i}\frac{\partial}{\partial x_i}\right]$$

We shall now show that, under certain conditions on V, models Σ_A and Σ_B are mixing.

Proposition 3.1. Suppose that:

(a) V is three times continuously differentiable.
(b) The first and second partial derivatives of grad V are bounded.
(c) $\exp(-\beta V) \in L^1(R^n)$.
(d) $(\partial V/\partial q_i)^2 \exp(-\beta V) \in L^1(R^n)$, $i = 1,..., n$.

Let the distribution $\bar{m}$ on R^{2n} be given by the probability density $N \exp[-\beta H(q, p)]$, where $N^{-1} = \int_X \exp[-\beta H(q, p)]\, dq\, dp$. Then $\bar{m}$ is a stationary distribution for Σ_A, and Σ_A is mixing with respect to $\bar{m}$.

Proposition 3.2. In addition to (a) and (b) of Proposition 3.1, let V satisfy:

(c′) $\int_{R^n} \exp[-2V(x)/\lambda^2]\, dx = N_\lambda^{-1} < \infty$.
(d′) $(\partial V/\partial x_i)^2 \exp(-2V/\lambda^2) \in L^1(R^n)$, $i = 1,..., n$.

Let the distribution $\bar{m}_\lambda$ on R^n be given by the probability density $N_\lambda \exp[-2V(x)/\lambda^2]$. Then $\bar{m}_\lambda$ is a stationary distribution for Σ_B, and Σ_B is mixing with respect to $\bar{m}_\lambda$.

In order to prove these propositions, the following results will be needed. The proofs of Propositions 3.3 and 3.4 can be found in Ref. 2, Part II, Par. 9 and Ref. 19, p. 160, while those of Lemmas 3.1 and 3.2 are given in the appendix.

Proposition 3.3. Let $Y(t) \in R^m$ be a Markovian diffusion process satisfying

$$dY(t) = a(Y(t))\, dt + b(Y(t))\, dw(t)$$

where $w(t)$ is a d-dimensional Wiener process, and the R^m-valued function $a(y)$ and the $m \times d$ matrix-valued function $b(y)$ satisfy the conditions of the existence and uniqueness theorem (see Ref. 1, p. 105) and have continuous, bounded first and second partial derivatives. Let $C^{(2)}(R^m)$ denote the continuous, bounded functions on R^m having continuous, bounded first and second partial derivatives, and define $\{T_t\}$ by Eq. (5), where $P(t, x, A)$ is the transition probability of $Y(t)$. Then for any $f \in C^{(2)}(R^m)$, $\tau < \infty$, the function $T_t f$ is continuous and bounded on $[0, \tau] \times R^m$ and has continuous, bounded first and second partial derivatives with respect to the $y_1,..., y_m$ and a continuous partial derivative with respect to t on this domain. Also,

$$\frac{\partial T_t f(y)}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{m} B_{ij}(y) \frac{\partial^2 T_t f(y)}{\partial y_i\, \partial y_j} + \sum_{i=1}^{m} a_i(y) \frac{\partial T_t f(y)}{\partial y_i}$$

$$\lim_{t \downarrow 0} T_t f(y) = f(y)$$

where $B(y)$ is the $m \times m$ matrix $b(y)b'(y)$.

Proposition 3.4. Let $\mathscr{L}$ be a family of functions on R^m, $m < \infty$, which is closed under addition, scalar multiplication, and bounded pointwise limits. If $\mathscr{L}$ contains $C^2_{\text{com}}(R^m)$, then $\mathscr{L}$ contains all bounded, Borel functions on R^m.

Lemma 3.1. Let the function $f(u, v)$ be locally integrable on $R^n \times R^d$ [i.e., $f 1_A \in L^1(R^{n+d})$ for any bounded subset A of R^{n+d}], and suppose that $\int_{R^n} f(u, v)\psi(u)\, du$ is v-independent almost everywhere (Lebesgue), for any $\psi \in C^2_{\text{com}}(R^n)$. Then f is v-independent almost everywhere.

Lemma 3.2. Let h be a locally integrable function on R^n such that, for some $i \leqslant n$, $\int_{R^n} h(u)(\partial\psi/\partial u_i)\, du = 0$ for all $\psi \in C^2_{\text{com}}(R^n)$. Then h is u_i-independent almost everywhere.

Proof of Proposition 3.1. We first show that the distribution $\bar{m}$ is stationary. Suppose that $f \in C^{(2)}(X)$, and let I be a bounded interval of R^+ containing $t \geqslant 0$. From (b), (d), and Proposition 3.3 we deduce the existence of $c < \infty$ such that, for all $s \in I$, $x \in X$,

$$\left|\frac{\partial T_s f(x)}{\partial s}\right| \leqslant c \sum_{i=1}^{n} \left[\frac{1}{2}\lambda^2 + \left|\frac{\partial V(q)}{\partial q_i}\right| + \left(1 + \frac{1}{2}\beta\lambda^2\right)|p_i|\right] \in L^1(X, \bar{m})$$

Hence we may differentiate under the integral sign and use (c), (d), and Proposition 3.3 to obtain

$$\frac{d}{dt}\int_X T_t f(x)\, d\bar{m}(x) = N \int_X G T_t f(x) \exp[-\beta H(q, p)]\, dq\, dp = 0$$

Let $\mathscr{L} = \{g \in B(X, \mathscr{B}) \colon \int_X T_t g(x)\, d\bar{m}(x) = \int_X g(x)\, d\bar{m}(x)$ for all $t \geqslant 0\}$. Now, $\mathscr{L}$ is closed under bounded pointwise limits, and, by the above, $C^2_{\text{com}}(X) \subset \mathscr{L}$. It follows from Proposition 3.4 that $1_A \in \mathscr{L}$ for any $A \in \mathscr{B}$, and hence that

$$\int_X P(t, x, A)\, d\bar{m}(x) = \int_X T_t 1_A(x)\, d\bar{m}(x) = \bar{m}(A) \quad \text{for all} \quad t \geqslant 0$$

Now by Chapter XIII, Par. 1, Theorem 1 of Ref. 20, Eq. (5) defines $\{T_t\}$ as a contraction semigroup of positive linear operators on the Hilbert space $\mathscr{H} = L^2(X, \bar{m})$, with norm $\| \ \|$ and inner product $\langle\ \rangle$. The domain of strong continuity of $\{T_t\}$ on $\mathscr{H}$ includes $C_0(X)$, and is norm-closed in $\mathscr{H}$ by general semigroup theory; hence it is the whole of $\mathscr{H}$. Let this semigroup have infinitesimal generator Q, with domain $\text{Dom}(Q)$.

Let $g \in C^{(2)}(X)$. Using (b)–(d) and Proposition 3.3, we can again differentiate under the integral sign and integrate by parts to obtain

$$(d/dt)\|T_t g\|^2 = -\lambda^2 \sum_{i=1}^{n} \|D_i T_t g\|^2$$

where D_i is the differential operator $\partial/\partial p_i$, defined on $C^{(2)}(X) \subset \mathscr{H}$. Thus $\|T_t g\|^2$ is decreasing, bounded below, and absolutely continuous, and so there exists a sequence $(t_\sigma) \uparrow \infty$ such that

$$\lim_{\sigma\to\infty} \|D_i T_{t_\sigma} g\| = 0, \qquad i = 1,..., n \tag{6}$$

Using the terminology of Bruck[21] we call such a sequence a $(*)$-sequence. The set $\{T_t g: t \geqslant 0\}$ is bounded and hence (see Ref. 20, p. 126) weakly sequentially precompact, so that by passing to a subsequence we may assume that

$$\underset{\sigma\to\infty}{\text{weak-lim}}\, T_{t_\sigma} g = \gamma \qquad \text{for some} \quad \gamma \in \mathscr{H}$$

Define W_i on the domain $C^2_{\text{com}}(X)$ by

$$W_i f = \beta p_i f - \partial f/\partial p_i$$

Then for each $i = 1,..., n$, D_i and W_i are adjoint, and so the adjoint operator D_i^* exists and extends W_i. Let $\psi \in C^2_{\text{com}}(X)$. For each i,

$$\begin{aligned} \langle \gamma, W_i\psi\rangle &= \lim_{\sigma\to\infty} \langle T_{t_\sigma} g, D_i^*\psi\rangle \\ &= \lim_{\sigma\to\infty} \langle D_i T_{t_\sigma} g, \psi\rangle = 0 \qquad \text{by} \quad (6) \end{aligned} \tag{7}$$

Let $\psi_1 \in C^2_{\text{com}}(R^n)$. Then $\bar{\gamma}(p) = \int_{R^n} \{\exp[-\beta V(q)]\}\gamma(q, p)\psi_1(q)\, dq$ is defined outside a ψ_1-independent null set and locally integrable on R^n, and it follows from (7) that

$$\int_{R^n} \bar{\gamma}(p) \frac{\partial}{\partial p_i}\left\{\left[\exp\left(-\frac{1}{2}\beta|p|^2\right)\right]\psi_2(p)\right\} dp = 0$$

for any $\psi_2 \in C^2_{\text{com}}(R^n)$ and $i = 1,..., n$. Hence by Lemma 3.2, $\bar{\gamma}$ is a constant. Since ψ_1 was arbitrary, it now follows from Lemma 3.1 that $\gamma = \gamma(q)$ is independent of the p coordinates.

As in Ref. 21, call a sequence $(s_\sigma') \subset R^+$ an almost-$(*)$-sequence if $(s_\sigma') \uparrow \infty$ and there exists a $(*)$-sequence (t_σ') such that $\lim_{\sigma\to\infty}(s_\sigma' - t_\sigma') = 0$. Let $0 < \tau < \min(s_1', t_1')$. Since

$$\|(d/dt)T_t g\| = \|T_{t-\tau} Q T_\tau g\| \leqslant \|Q T_\tau g\| \qquad \text{for any} \quad t \geqslant \tau$$

we have

$$\|T_{s_\sigma'} g - T_{t_\sigma'} g\| \leqslant |s_\sigma' - t_\sigma'|\, \|Q T_\tau g\| \to 0 \qquad \text{as} \quad \sigma \to \infty$$

Hence if either $\text{weak-lim}_{\sigma\to\infty} T_{s_\sigma'} g$ or $\text{weak-lim}_{\sigma\to\infty} T_{t_\sigma'} g$ exists, then both limits exist and are equal. Moreover, it has been shown that this common limit, if it exists, is a function of the q coordinates only.

We have demonstrated the existence of a (*)-sequence (t_σ) such that weak-$\lim_{\sigma\to\infty} T_{t_\sigma} g = \gamma$, a p-independent function. Let $t > 0$. Then weak-$\lim_{\sigma\to\infty} T_{t+t_\sigma} g = T_t\gamma$. Since $(t + t_\sigma) \uparrow \infty$, it is easy to show[21] that $(t + t_\sigma)$ has an almost-(*)-subsequence (s_σ). Clearly, weak-$\lim_{\sigma\to\infty} T_{s_\sigma} g = T_t\gamma$, and it follows from the remarks of the previous paragraph that $T_t\gamma$ is a function of the q coordinates only.

Consider next the dual semigroup $\{T_t^*\}$ on $\mathscr{H}$. Now, $\{T_t^*\}$ is weakly continuous, and hence also strongly continuous, since $\mathscr{H}$ is separable. Let its infinitesimal generator be Z. If $f \in \mathrm{Dom}(Z)$, $t \geqslant 0$, then $T_t^* f \in \mathrm{Dom}(Z)$, and for any $h \in \mathrm{Dom}(Q)$,

$$\begin{aligned}\langle ZT_t^*f, h\rangle &= \lim_{\epsilon\to 0} \epsilon^{-1}\langle (T_{t+\epsilon}^* - T_t^*)f, h\rangle \\ &= \lim_{\epsilon\to 0} \epsilon^{-1}\langle f, (T_{t+\epsilon} - T_t)h\rangle = \langle T_t^*f, Qh\rangle\end{aligned}$$

Thus $T_t^*f \in \mathrm{Dom}(Q^*)$ and $Q^*T_t^*f = ZT_t^*f$.

Note that for any $\varphi \in C^2_{\mathrm{com}}(X)$ and $t \geqslant 0$,

$$\langle \varphi, QT_t g\rangle = (d/dt)\langle \varphi, T_t g\rangle = \langle (G_1 + G_2)\varphi, T_t g\rangle \tag{8}$$

since we may use Proposition 3.3 and the fact that φ has compact support to differentiate under the integral and integrate by parts. Hence, since $C^2_{\mathrm{com}}(X) \subseteq \mathrm{Dom}(Z)$,

$$\begin{aligned}(d/dt)\langle \varphi, T_t\gamma\rangle &= \langle ZT_t^*\varphi, \gamma\rangle = \langle Q^*T_t^*\varphi, \gamma\rangle \\ &= \lim_{\sigma\to\infty} \langle \varphi, QT_{t+t_\sigma} g\rangle \\ &= \lim_{\sigma\to\infty} \langle (G_1 + G_2)\varphi, T_{t+t_\sigma} g\rangle \qquad \text{by (8)} \\ &= \langle (G_1 + G_2)\varphi, T_t\gamma\rangle = \langle G_2\varphi, T_t\gamma\rangle\end{aligned}$$

since $T_t\gamma$ is p-independent. Let $\varphi(q, p) = \varphi_1(q)\varphi_2(p)$, where $\varphi_1, \varphi_2 \in C^2_{\mathrm{com}}(R^n)$ are chosen arbitrarily subject to

$$\int_{R^n} [\exp(-\tfrac{1}{2}\beta|p|^2)]\varphi_2(p)\, dp = 0$$

Then for all $t \geqslant 0$, $\langle \varphi, T_t\gamma\rangle = 0$, and so

$$\begin{aligned}0 &= \langle G_2\varphi, T_t\gamma\rangle \\ &= N \sum_{i=1}^{n} \left\{ \int_{R^n} p_i \exp(-\tfrac{1}{2}\beta|p|^2)\, \varphi_2(p)\, dp \right\} \\ &\quad \times \left\{ \int_{R^n} \exp[-\beta V(q)]\, T_t\gamma(q) \beta\varphi_1 \left[\frac{\partial V}{\partial q_i} - \frac{\partial \varphi_1(q)}{\partial q_i}\right] dq \right\}\end{aligned}$$

It follows from the arbitrariness of φ_2 that

$$\int_{R^n} \gamma(q) \frac{\partial}{\partial q_i} \{\exp[-\beta V(q)]\varphi_1(q)] \, dq = 0$$

for $i = 1,..., n$ and any $\varphi_1 \in C^2_{\text{com}}(R^n)$. By (a) and Lemma 3.2, γ is therefore a constant. From the invariance of $\bar{m}$ we deduce that

$$\gamma = \langle g, 1 \rangle = \int_X g(x) \, d\bar{m}(x)$$

Finally, suppose that weak-$\lim_{t\to\infty} T_t g \neq \langle g, 1\rangle$. Then, by weak sequential precompactness of $\{T_t g: t \geqslant 0\}$, there must exist a sequence $(u_\sigma) \subset R^+$, $(u_\sigma) \uparrow \infty$, for which weak-$\lim_{\sigma\to\infty} T_{u_\sigma} g = \eta \neq \langle g, 1\rangle$. Then[21] (u_σ) has an almost-(*)-subsequence (s_σ'), and it follows that there exists a (*)-sequence (t_σ') for which weak-$\lim_{\sigma\to\infty} T_{t_\sigma'} g = \eta$. However, we have proved that for such a sequence we must have weak-$\lim_{\sigma\to\infty} T_{t_\sigma'} g = \langle g, 1\rangle$. This contradiction shows that weak-$\lim_{t\to\infty} T_t g = \langle g, 1\rangle$ for any $g \in C^{(2)}(X)$, and hence by density also for any $g \in \mathscr{H}$.

Density arguments now show that

$$\lim_{t\to\infty} \int_X T_t f(x) h(x) \, dx = \int_X f(x) \, d\bar{m}(x) \int_X h(x) \, dx$$

for any $f \in C_0(X)$, $h \in L^1(X)$, and the conclusion of Proposition 3.1 follows.

Proof of Proposition 3.2. Since the proof of this proposition is very similar to, although slightly less involved than, that of Proposition 3.1, we give only a brief outline of the main stages involved.

We now take $X = R^n$, and let G be the differential operator corresponding to Σ_B. Using (b), (c′), and (d′), $\bar{m}_\lambda$ is shown to be a stationary distribution for Σ_B by an argument similar to that used to prove invariance of $\bar{m}$ for Σ_A, and $\{T_t\}$ is defined as a strongly continuous contraction semigroup with generator Q on the Hilbert space $\mathscr{H} = L^2(X, \bar{m}_\lambda)$. Proceeding as in the proof of Proposition 3.1, and using (b) and (d′), we find that for $g \in C^{(2)}(X)$,

$$(d/dt)\|T_t g\|^2 = -\lambda^2 \sum_{i=1}^{n} \|D_i T_t g\|^2$$

where D_i is now the differential operator $\partial/\partial x_i$. The existence of a sequence $(t_\sigma) \uparrow \infty$ such that $\lim_{\sigma\to\infty} \|D_i T_{t_\sigma} g\| = 0$, $i = 1,..., n$, and weak-$\lim_{\sigma\to\infty} T_{t_\sigma} g = \gamma \in \mathscr{H}$ follows, and we can show that for any $\psi \in C^2_{\text{com}}(X)$ and $i = 1,..., n$, $\langle \gamma, W_i \psi\rangle = 0$, where now

$$W_i \psi = \frac{2}{\lambda^2} \frac{\partial V}{\partial x_i} \psi - \frac{\partial \psi}{\partial x_i}$$

An application of Lemma 3.2 leads to the conclusion that γ is a constant, and it is clear from the invariance of $\bar{m}_\lambda$ that this constant must be $\langle g, 1\rangle = \int_X g(x)\, d\bar{m}_\lambda(x)$.

The proof that weak-$\lim_{t\to\infty} T_t g = \langle g, 1\rangle$, and hence that Σ_B is mixing with respect to $\bar{m}_\lambda$, now follows almost exactly as in the case of Σ_A.

3.2. A Non-Markovian Model

Although it is not Markovian, model Σ_C can nevertheless also be shown to have the mixing property.

Proposition 3.5. Let $V: R^n \to R$ satisfy conditions (a)–(d) of Proposition 3.1, and let the distribution $\bar{m}$ on R^{2n} be defined as in that proposition. Then $\bar{m}$ is stationary for Σ_C, and Σ_C is mixing with respect to $\bar{m}$.

Proof. By increasing the dimension of the phase space, we shall recast the problem in a Markovian form. The proof will then proceed in a way similar to that of Proposition 3.1.

Define $r^\lambda(t) \in R^n$ by

$$r^\lambda(t) = -\lambda^2 c \int_0^t e^{-\alpha(t-s)} p^\lambda(s)\, ds + \lambda c \int_{-\infty}^t e^{-\alpha(t-s)}\, dw(s)$$

where $c = 2\alpha\beta^{-1}$. Then Eqs. (4) can be written

$$\begin{aligned} dq^\lambda(t) &= p^\lambda(t)\, dt \\ dp^\lambda(t) &= [-\operatorname{grad} V(q^\lambda(t)) + r^\lambda(t)]\, dt \\ dr^\lambda(t) &= [-\lambda^2 c p^\lambda(t) - \alpha r^\lambda(t)]\, dt + \lambda c\, dw(t) \end{aligned} \tag{9}$$

Equations (9) define a homogeneous diffusion process

$$Y^\lambda(t) = (q^\lambda(t), p^\lambda(t), r^\lambda(t))$$

on R^{3n}. Let $\{T_t\}$ be the corresponding contraction semigroup on the space of bounded, measurable functions on R^{3n} (see Section 3.1). The restriction of $\{T_t\}$ to $C_0(R^{3n})$ is a continuous semigroup whose generator is an extension of the differential operator G defined on $C^2_{\text{com}}(R^{3n})$ by

$$\begin{aligned} G &= \frac{1}{2}\lambda^2 c^2 \left(\exp \frac{\alpha|r|^2}{\lambda^2 c^2}\right) \sum_{i=1}^n \frac{\partial}{\partial r_i}\left[\left(\exp \frac{-\alpha|r|^2}{\lambda^2 c^2}\right)\frac{\partial}{\partial r_i}\right] \\ &\quad - \sum_{i=1}^n \left[\frac{\partial V(q)}{\partial q_i}\frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial q_i}\right] - \sum_{i=1}^n \left[\lambda^2 c p_i \frac{\partial}{\partial r_i} - r_i \frac{\partial}{\partial p_i}\right] \\ &= G_1 - G_2 - G_3, \quad \text{say} \end{aligned}$$

Let $\bar{\mu}_\lambda$ be the distribution on R^{3n} that corresponds to the probability density const $\times \exp\{-\beta[V(q) + \frac{1}{2}|p|^2 + \alpha|r|^2/\beta\lambda^2 c^2]\}$. Proceeding as in the proof of Proposition 3.1, we show that $\bar{\mu}_\lambda$ is a stationary distribution for

$Y^\lambda(t)$, and that $\{T_t\}$ is a strongly continuous semigroup, whose generator we again call Q, on the Hilbert space $\mathscr{H} = L^2(R^{3n}, \bar{\mu}_\lambda)$. Let $C^{(2)}(R^{3n})$ be defined as in Proposition 3.3, and let $\| \ \|$ and $\langle \ \rangle$ denote, respectively, the norm and the inner product in $\mathscr{H}$. For $g \in C^{(2)}(R^{3n})$,

$$(d/dt)\|T_t g\|^2 = -\lambda^2 c^2 \sum_{i=1}^{n} \|D_i T_t g\|^2$$

where D_i is now the differential operator $\partial/\partial r_i$. We can find as before a sequence $(t_\sigma) \uparrow \infty$ such that $\lim_{\sigma\to\infty} \|D_i T_{t_\sigma} g\| = 0$, $i = 1,..., n$, and weak-$\lim_{\sigma\to\infty} T_{t_\sigma} g = \gamma \in \mathscr{H}$, where γ is independent of the r coordinates. By an argument paralleling that used in Proposition 3.1, we now prove that, for any $\varphi \in C^2_{\mathrm{com}}(R^{3n})$,

$$\langle (G_1 + G_2 + G_3)\varphi, \gamma \rangle = 0$$

and hence that γ is a constant element of $\mathscr{H}$. It follows from the stationarity of $\bar{\mu}_\lambda$ that $\gamma = \langle g, 1 \rangle$.

The distribution of $(q^\lambda(t), p^\lambda(t))$ in R^{2n} is $m_{\lambda,t}$. Let that of $Y^\lambda(t)$ in R^{3n} be $\mu_{\lambda,t}$. Since $r^\lambda(0)$ is Gaussian, $m_{\lambda,0}$ is given by a probability density if and only if the same is true of $\mu_{\lambda,0}$, and in this case it can be shown as in Proposition 3.1 that $\lim_{t\to\infty} \mu_{\lambda,t} = \bar{\mu}_\lambda$ in the weak topology of measures. Integrating out over the r coordinates yields the desired mixing result for Σ_C. QED

For a non-Markovian model of this form, we have therefore shown that the variables become distributed according to the Maxwell–Boltzmann distribution as $t \to \infty$ whenever their initial distribution is given by a probability density. This approach to equilibrium occurs for arbitrary fixed, positive values of the coupling constant λ, and no weak coupling limit is required.

4. QUASIDETERMINISTIC BEHAVIOR

For $\lambda \geqslant 0$ let the stochastic process $x^\lambda(t)$ satisfy Eq. (1) with initial condition $x^\lambda(0) = x_0$. Let Σ be a stochastic system whose evolution is described by $x^\lambda(t)$. In physical applications the interaction parameter λ sometimes turns out to be very small in some appropriate sense, being a negative power of a macroscopic quantity,[3] and in this case it is clearly

[3] For example, in the case of a Brownian particle with velocity $v(t)$ acted on by a frictional force $F(v(t))$, we have the Langevin equation

$$m\, dv(t) = F(v(t))\, dt + dw(t)$$

where $w(t)$ is a Wiener process, and m ($\gg 1$) is the ratio of the mass of the Brownian particle to that of an atom in the host liquid. On rescaling the time variable to $\tau = t/m$, the above equation becomes

$$dv(\tau) = F(v(\tau))\, d\tau + m^{-1/2}\, dw(\tau)$$

and hence provides an example of Eq. (1) with $\lambda = m^{-1/2} \ll 1$.

desirable to ascertain whether $x^\lambda(t)$ exhibits only small fluctuations about $x^0(t)$, where $x^0(t)$ is the solution of the deterministic equation

$$dx^0(t) = \mathscr{F}^0(x^0, t)\,dt; \qquad x^0(0) = x_0$$

Definitions. (a) Σ is partially quasideterministic (pqd) if $\lim_{\lambda\to 0} x^\lambda(t) = x^0(t)$ in probability, uniformly on bounded time intervals.

(b) Σ is quasideterministic (qd) if $\lim_{\lambda\to 0} x^\lambda(t) = x^0(t)$ in mean square (and hence also in probability), uniformly for $t \geqslant 0$.

In this section it will be proved that the pqd property holds under very general conditions, being satisfied in particular by each of the models Σ_A, Σ_B, and Σ_C. In addition, Σ_B will be shown to exhibit qd behavior if and only if the corresponding causal evolution $x^0(t)$ is adequately stable with respect to small changes in the initial conditions.

Proposition 4.1. Consider the equation for $x^\lambda(t)$ in R^n, $1 \leqslant n < \infty$, $t \geqslant 0$:

$$dx^\lambda(t) = F(x^\lambda(t))\,dt + \lambda\,dz(t); \qquad x^\lambda(0) = x_0 \tag{10}$$

where $\lambda \geqslant 0$, $x_0 \in R^n$, $z(t)$ is an R^n-valued stochastic process whose sample paths are continuous with probability 1, and F: $R^n \to R^n$ satisfies a Lipschitz condition with constant k. Equation (10) has a unique solution which is defined for all $t \geqslant 0$ and is continuous with probability 1. The corresponding stochastic system Σ is pqd.

Proposition 4.2. Consider the equations for $x^\lambda(t) = (q^\lambda(t), p^\lambda(t))$ in R^{2n}, $1 \leqslant n < \infty$,

$$\begin{aligned} dq^\lambda(t)/dt &= p^\lambda(t) \\ dp^\lambda(t)/dt &= F(q^\lambda(t)) - \lambda^2 \int_0^t K(t-s) p^\lambda(s)\,ds + \lambda f(t) \\ x^\lambda(0) &= x_0 = (q_0, p_0) \end{aligned} \tag{11}$$

where $\lambda \geqslant 0$, $x_0 \in R^{2n}$, F is as in Proposition 4.1, $f(t)$ is an R^n-valued stochastic process whose sample paths are continuous with probability 1, and K: $R^+ \to R$ is an L^1-class function. Equations (11) have a unique solution, which is defined for all $t \geqslant 0$ and continuous with probability 1, and the stochastic model Σ described by $x^\lambda(t)$ is pqd.

Corollary 4.1. Each of the models Σ_A, Σ_B, and Σ_C is pqd.

Proof of Proposition 4.1. The existence, uniqueness, and continuity of the solution of (10) are proved by the Picard method as in Chapter 8 of Ref. 15. Let $\tau > 0$, $\gamma > k$, and define χ to be the complete metric space of continuous functions φ: $[0, \tau] \to R^n$ such that $\varphi(0) = x_0$, with norm $\|\varphi\|_\chi =$

$\sup_{0 \leqslant s \leqslant \tau} e^{-\gamma s}|\varphi(s)|$. Also, with z a fixed, continuous function of t, define $m(\tau) = \sup_{0 \leqslant s \leqslant \tau}|z(s) - z(0)| < \infty$. For $s \in [0, \tau]$, $|x^\lambda(s) - x^0(s)| \leqslant k \int_0^s |x^\lambda(u) - x^0(u)|\, du + \lambda|z(s) - z(0)|$ and hence

$$\|x^\lambda - x^0\|_\chi \leqslant k\|x^\lambda - x^0\|_\chi \sup_{0 \leqslant s \leqslant \tau} e^{-\gamma s} \int_0^s e^{\gamma u}\, du + \lambda m(\tau)$$

so that

$$\|x^\lambda - x^0\|_\chi \leqslant [\gamma/(\gamma - k)]\lambda m(\tau)$$

Thus for any $t \in [0, \tau]$, $|x^\lambda(t) - x^0(t)| \leqslant e^{\gamma\tau}[\gamma/(\gamma - k)]\lambda m(\tau) \to 0$ as $\lambda \to 0$, uniformly for $t \in [0, \tau]$. By the assumption about the sample paths of $z(t)$ it follows that $\lim_{\lambda \to 0} x^\lambda(t) = x^0(t)$, uniformly over $t \in [0, \tau]$, the limit holding with probability 1 and hence also in probability. Since τ was arbitrary, it follows that Σ is pqd.

Proof of Proposition 4.2. The existence, uniqueness, and continuity of the solution of (11) are again proved by the Picard method. To prove the pqd property, we write

$$q^\lambda(t) = q_0 + \int_0^t p^\lambda(s)\, ds$$

$$p^\lambda(t) = p_0 + \int_0^t F(q^\lambda(s))\, ds - \lambda^2 \int_0^t G(t - s)p^\lambda(s)\, ds + \lambda \int_0^t f(s)\, ds$$

where $G(u) = \int_0^u K(y)\, dy$. For a fixed continuous sample path $f(t)$, and $\tau > 0$, let $\int_0^\tau |f(s)|\, ds = m(\tau) < \infty$, and define $\|x\|_\sim = \max(|q|, |p|)$ for $x = (q, p) \in R^{2n}$. Then, for all $t \in [0, \tau]$,

$$\|x^\lambda(t) - x^0(t)\|_\sim \leqslant \lambda m(\tau) + (1 + k + \lambda^2\|K\|_1) \int_0^t \|x^\lambda(s) - x^0(s)\|_\sim\, ds$$

so that by the Bellman–Gronwall lemma (see Ref. 1, p. 107; Ref. 2, p. 41),

$$\begin{aligned}\|x^\lambda(t) - x^0(t)\|_\sim &\leqslant \lambda m(\tau)\{1 + (1 + k + \lambda^2\|K\|_1) \\ &\quad \times \int_0^t \exp[(1 + k + \lambda^2\|K\|_1)(t - s)]\, ds\} \\ &\leqslant \lambda m(\tau) \exp[(1 + k + \lambda^2\|K\|_1)\tau] \qquad \text{for all} \quad t \in [0, \tau]\end{aligned}$$

Hence $|x^\lambda(t) - x^0(t)| \leqslant 2^{1/2}\|x^\lambda(t) - x^0(t)\|_\sim \to 0$ as $\lambda \to 0$, uniformly for $t \in [0, \tau]$. QED

We turn next to an investigation of conditions under which the more restrictive qd property will hold.

Proposition 4.3. Let the R^n-valued stochastic process $x^\lambda(t)$ satisfy

$$dx^\lambda(t) = F(x^\lambda(t))\,dt + \lambda\,dw(t); \qquad x^\lambda(0) = x_0 \tag{12}$$

where $w(t)$ is an n-dimensional Wiener process, and $F: R^n \to R^n$ is continuously differentiable with derivative DF satisfying (i) $\|DF(x)\| \leqslant M < \infty$ for all $x \in R^n$, (ii) $DF(x) \leqslant -AI$ for all $x \in R^n$ and some $A > 0$, where I is the identity map on R^n and $\|\ \|$ is the operator norm on the space of linear maps of R^n to R^n. Then the corresponding stochastic system Σ is qd.

The following lemmas, which are special cases of results proved in Part II, Pars. 8 and 13 of Ref. 2, will be needed for the proof of Proposition 4.3.

Lemma 4.1. With $x^\lambda(t)$ defined by (12), let $z^\lambda(t) \in R^n$ be the solution of

$$dz^\lambda(t) = DF(x^\lambda(t))z^\lambda(t)\,dt + dw(t); \qquad z^\lambda(0) = 0$$

Then the mean square derivative of $x^\lambda(t)$ with respect to λ exists and is given by $z^\lambda(t)$ for all $t \geqslant 0$.

Lemma 4.2. $\mathscr{E}(|z^\lambda(t)|^2) \leqslant n/(2A)$ for all $t \geqslant 0$, where $\mathscr{E}$ denotes the expectation with respect to the underlying probability space.

Proof of Proposition 4.3. Using Lemma 4.1 and the mean value theorem,

$$[\mathscr{E}(|x^\lambda(t) - x^0(t)|^2)]^{1/2} \leqslant \lambda \sup_{0 \leqslant \epsilon \leqslant \lambda} [\mathscr{E}(|z^\epsilon(t)|^2)]^{1/2}$$

so that by Lemma 4.2,

$$\mathscr{E}(|x^\lambda(t) - x^0(t)|^2) \leqslant \lambda^2 n/(2A)$$

It follows that Σ is qd. QED

As a corollary of Proposition 4.3, we see that model Σ_B will be qd provided that the corresponding causal evolution is highly stable with respect to changes in the initial conditions. The following example shows that this strong stability requirement is also necessary in order to ensure qd behavior of Σ_B.

Example. Let $X = R$, and suppose that $V: X \to R$ has the form shown in Fig. 1 and satisfies suitable regularity conditions. Let $x^\lambda(t)$ satisfy Eq. (3) with $n = 1$ and $F = -dV/dx$. By taking V to be a Liapunov function for the deterministic trajectory, we see that $x^0(t)$, if initially in the well at A, will remain there for all time and exhibit stable behavior with $\lim_{t\to\infty} x^0(t) = x_1$. However, it was shown in Section 3 that the distribution $m_{\lambda,t}$ of $x^\lambda(t)$ will tend to $\bar{m}_\lambda$ as $t \to \infty$, where $\bar{m}_\lambda$ is the stationary distribution given by the probability density const $\times \exp[-2V(x)/\lambda^2]$. It is easy to prove (see Ref. 17) that, for V as shown, $\lim_{\lambda\to 0} \bar{m}_\lambda = \delta_{x_2}$ in the weak topology of measures,

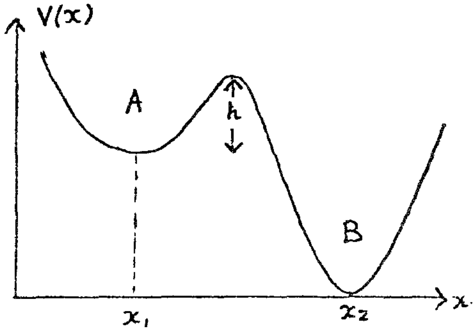

Fig. 1

where δ_{x_2} is the delta distribution concentrated at x_2. Thus this particular model is not qd, since the dispersion of $x^\lambda(t) - x^0(t)$ does not tend to zero uniformly in $t \geqslant 0$ as $\lambda \to 0$, owing to the ability of the fluctuating $x^\lambda(t)$ to diffuse over the barrier between the wells A and B.[4]

This example shows that the occurrence of even arbitrarily weak stochastic forces may lead to a drastic departure from the deterministic law. It also demonstrates that the order of taking limits of $m_{\lambda,t}$ as $\lambda \to 0$ and $t \to \infty$ may be critical. For, let $m_{\lambda,0}$ be given by a probability density that is concentrated inside the well A. Then, by the pqd property of the model, and the given form of V, we have $\lim_{t\to\infty} \lim_{\lambda\to 0} m_{\lambda,t} = \delta_{x_1}$, whereas

$$\lim_{\lambda\to 0} \lim_{t\to\infty} m_{\lambda,t} = \delta_{x_2},$$

showing that the long-time and weak coupling limits may not in general be interchanged.

The presence of fluctuations completely changes the stability properties of the system. For the causal evolution several stable stationary states, associated with distinct local minima of V, may coexist, whereas when the stochastic forces are taken into account there is a unique steady state, which is approached in time by all other states. Any local instabilities of the causal evolution now disappear, manifesting themselves only in the detailed form of the stationary state.

Gaussian Approximation to $x^\lambda(t)$. It has been shown in Proposition 4.3 that, providing (i) and (ii) are satisfied, $x^\lambda(t) - x^0(t)$ is $O(\lambda)$ in mean square, uniformly for $t \geqslant 0$. Thus $x^\lambda(t)$ can be approximated for all positive times by the deterministic trajectory $x^0(t)$, to first order in the small parameter λ.

[4] It should nevertheless be remarked that since[16] the probability per unit time that $x^\lambda(t)$ will escape from A over the barrier of height h is proportional to $\exp(-\beta h/\lambda^2)$, where β is the inverse temperature, Σ may, if h is very large or λ very small, behave in a nearly deterministic manner for a long period of time.

If in addition the function F is twice continuously differentiable with bounded second derivative, we can again use results of Ref. 2, Part II, Pars. 8 and 13 on the differentiation with respect to a parameter of stochastic processes and the boundedness of their moments to show that, for any $\lambda \geqslant 0$, the second derivative of $x^\lambda(t)$ with respect to λ exists and is bounded in mean square for all $t \geqslant 0$. It then follows that $x^\lambda(t) - x^0(t) - \lambda y(t)$ is $O(\lambda^2)$ in mean square, uniformly for $t \geqslant 0$, where $y(t) = z^0(t)$ is the Gaussian process given by

$$dy(t) = DF(x^0(t))y(t)\,dt + dw(t); \qquad y(0) = 0$$

$x^\lambda(t)$ may therefore be approximated to second order in λ by the Gaussian process $x^0(t) + \lambda y(t)$, uniformly for $t \geqslant 0$. Under the conditions stated above, this substantiates assertions made in Refs. 12 and 14.

5. CONCLUDING REMARKS

Certain finite-dimensional models, of types which are frequently used to describe the evolution of stochastic systems, have been studied. Conditions have been sought under which certain desirable properties of these models can be proved to hold; some of these properties have previously been treated in a nonrigorous manner or tacitly assumed to hold for physical systems.

By contrast with the usual treatments of steady-state probability densities for diffusion processes (see, e.g., Refs. 6, 7, and 11), no initial assumption is made in Section 3 that such a density will necessarily be smooth enough to satisfy the Fokker–Planck equation. Nor is consideration restricted, as in Ref. 11, to a class of well-behaved densities. Instead, for the process $x^\lambda(t)$ describing Σ_A or Σ_B it is proved that there is a unique stationary probability density in phase space which is approached by any other density as $t \to \infty$, and that for Σ_A this density corresponds to the Maxwell–Boltzmann distribution of thermal equilibrium.

Most of the existing results on the approach to equilibrium of nonlinear, non-Markovian systems have been obtained in the limit as $\lambda \to 0$, $t \to \infty$, $\tau = \lambda^2 t \in [0, \tau_0]$, $\tau_0 < \infty$. This (van Hove) limit has in particular been employed in rigorous treatments of open systems by means of generalized master equations[22] and of analogous problems formulated in terms of Banach space-valued stochastic processes.[23,24] It does not generally hold uniformly over all $\tau_0 \geqslant 0$. By contrast, no weak coupling limit $\lambda \to 0$ is used in our treatment of the non-Markovian model Σ_C, whose variables $x^\lambda(t)$ are shown to become distributed according to the Maxwell–Boltzmann distribution as $t \to \infty$ whenever their initial distribution is given by a probability density, and for any fixed $\lambda > 0$.[5]

[5] A similar result is obtained in Ref. 25 for a particular class of quantum systems.

While the property that we have defined as "pqd" is satisfied very generally, results of Section 4 show that a stochastic model will have the more restrictive "qd" property if and only if the corresponding deterministic trajectory, obtained by neglecting the fluctuating forces, shows a strong stability with respect to small perturbations of the initial conditions. We refer again to the example of Section 4 for an important illustration of the way in which, when this strong causal stability is lacking, a model that is pqd may nevertheless fail to be qd.

APPENDIX

Proof of Lemma 3.1. We first note that the sigma-field of Borel sets $\mathscr{B}^d$ on R^d is generated by the semiring $\mathscr{P}^d$ of half-open intervals, where $I \in \mathscr{P}^d$ has the form $I = \{v \in R^d: a_i < v_i \leqslant b_i, i = 1,..., d\}$ for some $a, b \in R^d$. Lebesgue measure is the unique extension to $\mathscr{B}^d$ of the set function $\mu: \mathscr{P}^d \to R^+$ defined by $\mu(I) = \prod_{i=1}^d (b_i - a_i)$. If the function g is locally integrable on R^d and satisfies

$$\int_I g(v)\, dv = \int_J g(v)\, dv \tag{A1}$$

for any $I, J \in \mathscr{P}^d$ with $\mu(I) = \mu(J) < \infty$, then

$$\int_I g(v)\, dv = \Phi(\mu(I)) \qquad \text{for some} \quad \Phi: R^+ \to R$$

Φ, being linear and continuous, has the form $\Phi(\mu(I)) = c\mu(I)$ for some $c \in R$, and it follows that $g(v) = c$ almost everywhere (Lebesgue). Moreover, by the local integrability of g, it will suffice to assume (A1) for each of the countable set of pairs of intervals (I, J) having rational end points and satisfying $\mu(I) = \mu(J) < \infty$.

Now for $I, J \in \mathscr{P}^d$ with $\mu(I) = \mu(J) < \infty$ we have, interchanging the order of integration by Fubini's theorem,

$$\int_{R^n} \psi(u)\, du \left\{ \int_I f(u, v)\, dv - \int_J f(u, v)\, dv \right\} = 0 \qquad \text{for any} \quad \psi \in C^2_{\text{com}}(R^n)$$

Hence for all u outside a Lebesgue null set,

$$\int_I f(u, v)\, dv = \int_J f(u, v)\, dv \tag{A2}$$

Taking a countable union of null sets, we obtain a null set N_1 such that for all u in the complement $N_1{}^c$ of N_1, (A2) holds for all (I, J) having rational end points and finite, equal Lebesgue measures. Since $f(u, v)$ is locally integrable for all u outside a null set N_2, it follows from the discussion at the beginning of the proof that, for $u \in (N_1 \cup N_2)^c, f(u, v) = f(u)$ is v-independent almost everywhere.

Proof of Lemma 3.2. Let $I = (a, b]$ and $J = (c, d]$ be elements of $\mathscr{P}^1$ with $b \leqslant c$ and $b - a = d - c$. Let χ be the function on R that is linear on I and J and satisfies

$$\begin{aligned} \chi(t) &= 0 \qquad &&\text{if} \quad t \leqslant a \quad \text{or} \quad t \geqslant d \\ &= b - a \qquad &&\text{if} \quad b \leqslant t \leqslant c \end{aligned}$$

Then there exists a sequence $\{\theta^{(m)}\}$ in $C^2_{\text{com}}(R)$ such that $|\theta^{(m)}|$ and $|d\theta^{(m)}/dt|$ are bounded independently of m, and for each $t \in R$

$$\lim_{m \to \infty} \theta^{(m)}(t) = \chi(t); \qquad \lim_{m \to \infty} d\theta^{(m)}(t)/dt = 1_I(t) - 1_J(t)$$

Let $\psi^{(m)}(u) = \varphi(\hat{u})\theta^{(m)}(u_i)$, where $\hat{u} = (u_1, ..., u_{i-1}, u_{i+1}, ..., u_n)$ and $\varphi \in C^2_{\text{com}}(R^{n-1})$. By hypothesis,

$$\int_{R^n} h(u)(\partial\psi^{(m)}/\partial u_i)\, du = 0$$

Using the arbitrariness of φ, we deduce the existence of a null set N of R^{n-1} such that, if $\hat{u} \in N^c$,

$$\int_{-\infty}^{\infty} h(u)(d\theta^{(m)}/du_i)\, du_i = 0, \qquad m = 1, 2, ...$$

Hence by the dominated convergence theorem

$$\int_I h(u)\, du_i = \int_J h(u)\, du_i \qquad \text{for all} \quad \hat{u} \in N^c$$

The proof that h is independent of u_i almost everywhere now follows as in Lemma 3.1, using the local integrability of h. QED

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